UNIVERSAL COVERING CALABI-YAU MANIFOLDS OF THE HILBERT SCHEMES OF N POINTS OF ENRIQUES SURFACES

TARO HAYASHI

ABSTRACT. Throughout this paper, we work over $\mathbb{C}$, and n is an integer such that $n \geq 2$. For an Enriques surface E, let $E^{[n]}$ be the Hilbert scheme of n points of E. By Oguiso and Schröer [8, Theorem 3.1], $E^{[n]}$ has a Calabi-Yau manifold X as the universal covering space, $\pi : X \to E^{[n]}$ of degree 2. The purpose of this paper is to investigate a relationship of the small deformation of $E^{[n]}$ and that of X (Theorem 1.1), the natural automorphism of $E^{[n]}$ (Theorem 1.2), and count the number of isomorphism classes of the Hilbert schemes of n points of Enriques surfaces which has X as the universal covering space when we fix one X (Theorem 1.3).

1. Introduction

Throughout this paper, we work over $\mathbb{C}$, and n is an integer such that $n \geq 2$. For an Enriques surface E, let $E^{[n]}$ be the Hilbert scheme of n points of E. By Oguiso and Schröer [8, Theorem 3.1], $E^{[n]}$ has a Calabi-Yau manifold X as the universal covering space, $\pi : X \to E^{[n]}$ of degree 2. The purpose of this paper is to investigate a relationship of the small deformation of $E^{[n]}$ and that of X (Theorem 1.1), the natural automorphism of $E^{[n]}$ (Theorem 1.2), and count the number of isomorphism classes of the Hilbert schemes of n points of Enriques surfaces which has X as the universal covering space when we fix one X (Theorem 1.3).

Small deformations of a smooth compact surface S induce that of the Hilbert scheme of n points of S by taking the relative Hilbert scheme. Let K be a $K3$ surface. By Beauville [1, page 779-781], a very general small deformation of $K^{[n]}$ is not isomorphic to the Hilbert scheme of n points of a $K3$ surface. On the other hand, by Fantechi [3, Theorems0.1 and 0.3], every small deformations of $E^{[n]}$ is induced by that of E. Since X is the universal covering of $E^{[n]}$, the small deformation of $E^{[n]}$ induces that of X. We consider a relationship of the small deformation of $E^{[n]}$ and that of X. Our first main result is following:

Theorem 1.1. *Let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and X the universal covering space of $E^{[n]}$. Then every small deformation of X is induced by that of $E^{[n]}$.*

Compare with the fact that a general small deformation of the universal covering $K3$ surface of E is not induced by that of E.

Next, we study the natural automorphisms of $E^{[n]}$. Any automorphism $f \in \mathrm{Aut}(S)$ induces an automorphism $f^{[n]} \in \mathrm{Aut}(S^{[n]})$. An automorphism $g \in \mathrm{Aut}(S^{[n]})$

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is called natural if there is an automorphism $f \in \mathrm{Aut}(S)$ such that $g = f^{[n]}$. When K is a $K3$ surface, the natural automorphisms of $K^{[n]}$ have been studied by Boissière and Sarti [2, Theorem 1]. They used the global Torelli theorem for $K3$ surfaces: an effective Hodge isometry α is induced by a unique automorphism β of $K3$ surface such that $\alpha = \beta^*$. Our second main result is the following theorem, similar to [2, Theorem 1] without the Torelli theorem for Enriques surfaces by using a result of Oguiso [7, Proposition 4, 4].

Theorem 1.2. *Let E be an Enriques surface, D_E the exceptional divisor of the Hilbert-Chow morphism $\pi_E : E^{[n]} \to E^{(n)}$, and $n \geq 2$. An automorphism f of $E^{[n]}$ is natural if and only if $f(D_E) = D_E$, i.e. $f^*(\mathcal{O}_{E^{[n]}}(D_E)) = \mathcal{O}_{E^{[n]}}(D_E)$.*

Finally, we compute the number of isomorphism class of the Hilbert schemes of n points of Enriques surfaces which have X as the universal covering space when we fixed one X.

Theorem 1.3. *Let E and E' be two Enriques surfaces, $E^{[n]}$ and $E'^{[n]}$ the Hilbert scheme of n points of E and E', X and X' the universal covering space of $E^{[n]}$ and $E'^{[n]}$, and $n \geq 3$. If $X \cong X'$, then $E^{[n]} \cong E'^{[n]}$, i.e. when we fix X, then there is just one isomorphism class of the Hilbert schemes of n points of Enriques surfaces such that they have it as the universal covering space.*

Our proof is based on Theorem 1.2 and the study of the action of the covering involutions on $H^2(X, \mathbb{C})$.

This is the result that is greatly different from the result of Ohashi (See [9, Theorem 0.1]) that, for any nonnegative integer l, there exists a $K3$ surface with exactly 2^{l+10} distinct Enriques quotients. In particular, there does not exist a universal bound for the number of distinct Enriques quotients of a $K3$ surface. Here we will call two Enriques quotients of a K3 surface distinct if they are not isomorphic to each other.

Remark 1.4. When *n=2,* I do not count the number of isomorphism classes of the Hilbert schemes of n points of Enriques surfaces which has X as the universal covering space when we fix one *X.*

2. Preliminaries

A $K3$ surface K is a compact complex surface with $K_K \sim 0$ and $H^1(K, \mathcal{O}_K) = 0$. An Enriques surface E is a compact complex surface with $H^1(E, \mathcal{O}_E) = 0$, $H^2(E, \mathcal{O}_E) = 0$, $K_E \not\sim 0$, and $2K_E \sim 0$. The universal covering of an Enriques surface is a $K3$ surface. A Calabi-Yau manifold X is an n-dimensional compact kähler manifold such that it is simply connected, there is no holomorphic k-form on X for $0 < k < n$ and there is a nowhere vanishing holomorphic n-form on X.

Let S be a nonsingular surface, $S^{[n]}$ the Hilbert scheme of n points of S, $\pi_S : S^{[n]} \to S^{(n)}$ the Hilbert-Chow morphism, and $p_S : S^n \to S^{(n)}$ the natural projection. We denote by D_S the exceptional divisor of π_S. Note that $S^{[n]}$ is smooth of $\dim_{\mathbb{C}} S^{[n]} = 2n$. Let Δ_S^n be the set of n-uples $(x_1, \ldots, x_n) \in S^n$ with at least two x_i's equal, S_*^n the set of n-uples $(x_1, \ldots, x_n) \in S^n$ with at most two x_i's equal. We put

$$S_*^{(n)} := p_S(S_*^n),$$
$$\Delta_S^{(n)} := p_S(\Delta_S^n),$$

$$\begin{aligned} S_*^{[n]} &:= \pi_S^{-1}(S_*^{(n)}), \\ \Delta_{S*}^n &:= \Delta_S^n \cap S_*^n, \\ \Delta_{S*}^{(n)} &:= p_S(\Delta_{S*}^n), \text{ and} \\ F_S &:= S^{[n]} \setminus S_*^{[n]}. \end{aligned}$$

Then we have $\mathrm{Blow}_{\Delta_{S*}^n} S_*^n/\mathcal{S}_n \simeq S_*^{[n]}$, F_S is an analytic closed subset, and its codimension is 2 in $S^{[n]}$ by Beauville [1, page 767-768]. Here $\mathcal{S}_n$ is the symmetric group of degree n which acts naturally on S^n by permuting of the factors.

Let E be an Enriques surface, and $E^{[n]}$ the Hilbert scheme of n points of E. By Oguiso and Schröer [8, Theorem 3.1], $E^{[n]}$ has a Calabi-Yau manifold X as the universal covering space $\pi : X \to E^{[n]}$ of degree 2. Let $\mu : K \to E$ be the universal covering space of E where K is a $K3$ surface, S_K the pullback of $\Delta_E^{(n)}$ by the morphism

$$\mu^{(n)} : K^{(n)} \ni [(x_1, \ldots, x_n)] \mapsto [(\mu(x_1), \ldots, \mu(x_n))] \in E^{(n)}.$$

Then we get a 2^n-sheeted unramified covering space

$$\mu^{(n)}|_{K^{(n)}\setminus S_K} : K^{(n)}\setminus S_K \to E^{(n)}\setminus\Delta_E^{(n)}.$$

Furthermore, let Γ_K be the pullback of S_K by natural projection $p_K : K^n \to K^{(n)}$. Since Γ_K is an algebraic closed set with codimension 2, then

$$\mu^{(n)} \circ p_K : K^n\setminus\Gamma_K \to E^{(n)}\setminus\Delta_E^{(n)}$$

is the $2^n n!$-sheeted universal covering space. Since $E^{[n]}\setminus D_E = E^{(n)}\setminus\Delta_E^{(n)}$ where $D_E = \pi_E^{-1}(\Delta_E^{(n)})$, we regard the universal covering space $\mu^{(n)} \circ p_K : K^n\setminus\Gamma_K \to E^{(n)}\setminus\Delta_E^{(n)}$ as the universal covering space of $E^{[n]} \setminus D_E$:

$$\mu^{(n)} \circ p_K : K^n\setminus\Gamma_K \to E^{[n]}\setminus D_E.$$

Since $\pi : X \setminus \pi^{-1}(D_E) \to E^{[n]} \setminus D_E$ is a covering space and $\mu^{(n)} \circ p_K : K^n \setminus \Gamma_K \to E^{[n]} \setminus D_E$ is the universal covering space, there is a morphism

$$\omega : K^n \setminus \Gamma_K \to X \setminus \pi^{-1}(D_E)$$

such that $\omega : K^n\setminus\Gamma_K \to X\setminus\pi^{-1}(D_E)$ is the universal covering space and $\mu^{(n)}\circ p_K = \pi \circ \omega$:

$$\begin{tikzcd} K^n \setminus \Gamma_K \arrow[r, "\omega"] \arrow[dr, "\mu^{(n)}\circ p_K"'] & X \setminus \pi^{-1}(D_E) \arrow[d, "\pi"] \\ & E^{[n]} \setminus D_E. \end{tikzcd}$$

We denote the covering transformation group of $\pi \circ \omega$ by:

$$G := \{g \in \mathrm{Aut}(K^n \setminus \Gamma_K) : \pi \circ \omega \circ g = \pi \circ \omega\}.$$

Then G is of order $2^n.n!$, since $\deg(\mu^{(n)} \circ p_K) = 2^n.n!$. Let σ be the covering involution of $\mu : K \to E$, and for

$$1 \le k \le n,\ 1 \le i_1 < \cdots < i_k \le n$$

we define automorphisms $\sigma_{i_1\ldots i_k}$ of K^n by following. For $x = (x_i)_{i=1}^n \in K^n$,

$$\textit{the } j\textit{-th component of } \sigma_{i_1\ldots i_k}(x) = \begin{cases} \sigma(x_j) & j \in \{i_1, \cdots, i_k\} \\ x_j & j \notin \{i_1, \cdots, i_k\}. \end{cases}$$

Then $\mathcal{S}_n \subset G$, and $\{\sigma_{i_1\ldots i_k}\}_{1\le k\le n,\ 1\le i_1<\ldots<i_k\le n} \subset G$. Let H be the subgroup of G generated by $\mathcal{S}_n$ and $\{\sigma_{ij}\}_{1\le i<j\le n}$.

Proposition 2.1. *G is generated by $\mathcal{S}_n$ and $\{\sigma_{i_1\ldots i_k}\}_{1\le k\le n,\ 1\le i_1<\ldots<i_k\le n}$. Moreover any element is of the form $s\circ t$ where $s\in\mathcal{S}_n$, $t\in\{\sigma_{i_1\ldots i_k}\}_{1\le k\le n,\ 1\le i_1<\ldots<i_k\le n}$.*

Proof. If $(s,t)=(s',t')$ for $s,s'\in\mathcal{S}_n$ and $t,t'\in\{\sigma_{i_1\ldots i_k}\}_{1\le k\le n,\ 1\le i_1<\ldots<i_k\le n}$, then we have $s=s'$ and $t=t'$ by paying attention to the permutation of component. As $|\mathcal{S}_n|=n!$, and $|\{\sigma_{i_1\ldots i_k}\}_{1\le k\le n,\ 1\le i_1<\ldots<i_k\le n}|=2^n$, G is generated by $\mathcal{S}_n$ and $\{\sigma_{i_1\ldots i_k}\}_{1\le k\le n,\ 1\le i_1<\ldots<i_k\le n}$. □

Proposition 2.2. $|H|=2^{n-1}.n!$.

Proof. H is generated by $\mathcal{S}_n$ and $\{\sigma_{ij}\}_{1\le i<j\le n}$. By paying attention to the permutation of component, we have $\sigma_i\notin H$ for all i. For arbitrary j, $(i,j)\circ\sigma_i\circ(i,j)=\sigma_j$. Since $\mathcal{S}_n\subset H$, and Proposition 2.1, we obtain $|G/H|=2$, i.e. $|H|=2^{n-1}.n!$. □

We put

$$K^n_{*\mu} := (\mu^n)^{-1}(E^n_*),$$

where $\mu^n : K^n \ni (x_i)_{i=1}^n \mapsto (\mu(x_i))_{i=1}^n \in E^n$. Recall that $\mu: K\to E$ the universal covering with σ the covering involution. We further put

$$T_{*\mu\,ij} := \{(x_l)_{l=1}^n \in K^n_{*\mu} : \sigma(x_i)=x_j\},$$

$$\Delta_{K*\mu\,ij} := \{(x_l)_{l=1}^n \in K^n_{*\mu} : x_i=x_j\},$$

$$T_{*\mu} := \bigcup_{1\le i<j\le n} T_{*\mu\,i,j},\ \text{and}$$

$$\Delta_{K*\mu} := \bigcup_{1\le i<j\le n} \Delta_{K*\mu\,i,j}.$$

By the definition of $K^n_{*\mu}$, H acts on $K^n_{*\mu}$, and by the definition of $\Delta_{K*\mu}$ and $T_{*\mu}$, we have $\Delta_{K*\mu}\cap T_{*\mu}=\emptyset$.

Lemma 2.3. *For $t\in H$ and $1\le i<j\le n$, if $t\in H$ has a fixed point on $\Delta_{K*\mu\,ij}$, then $t=(i,j)$ or $t=\mathrm{id}_{K^n}$.*

Proof. Let $t\in H$ be an element of H where there is an element $\tilde{x}=(\tilde{x}_i)_{i=1}^n\in\Delta_{K*\mu\,ij}$ such that $t(\tilde{x})=\tilde{x}$. By Proposition 2.1, for $t\in H$, there are two elements $\sigma_{i_1,\cdots,i_k}\in\{\sigma_{i_1\ldots i_k}\}_{1\le k\le n,\ 1\le i_1<\ldots<i_k\le n}$ and $(j_1,\cdots,j_l)\in\mathcal{S}_n$ such that

$$t=(j_1,\cdots,j_l)\circ\sigma_{i_1,\cdots,i_k}.$$

From the definition of $\Delta_{K*\mu\,ij}$, for $(x_l)_{l=1}^n\in\Delta_{K*\mu\,ij}$,

$$\{x_1,\ldots,x_n\}\cap\{\sigma(x_1),\ldots,\sigma(x_n)\}=\emptyset.$$

Suppose $\sigma_{i_1,\cdots,i_k}\neq\mathrm{id}_{K^n}$. Since $t(\tilde{x})=\tilde{x}$, we have

$$\{\tilde{x}_1,\ldots,\tilde{x}_n\}\cap\{\sigma(\tilde{x}_1),\ldots,\sigma(\tilde{x}_n)\}\neq\emptyset,$$

a contradiction. Thus we have $t=(j_1,\cdots,j_l)$. Similarly from the definition of $\Delta_{K*\mu\,ij}$, for $(x_l)_{l=1}^n\in\Delta_{K*\mu\,ij}$, if $x_s=x_t$ $(1\le s<t\le n)$, then $s=i$ and $t=j$. Thus we have $t=(i,j)$ or $t=\mathrm{id}_{K^n}$. □

Lemma 2.4. *For $t\in H$ and $1\le i<j\le n$, if $t\in H$ has a fixed point on $T_{*\mu\,ij}$, then $t=\sigma_{i,j}\circ(i,j)$ or $t=\mathrm{id}_{K^n}$.*

Proof. Let $t \in H$ be an element of H where there is an element $\tilde{x} = (\tilde{x}_i)_{i=1}^n \in T_{K*\mu\, ij}$ such that $t(\tilde{x}) = \tilde{x}$. By Proposition 2.1, for $t \in H$, there are two elements $\sigma_{i_1,\cdots,i_k} \in \{\sigma_{i_1\ldots i_k}\}_{1\le k\le n,\ 1\le i_1<\ldots<i_k\le n}$ and $(j_1,\cdots,j_l) \in \mathcal{S}_n$ such that

$$t = (j_1,\cdots,j_l)\circ\sigma_{i_1,\cdots,i_k}.$$

Since $(j,j+1)\circ\sigma_{i,j}\circ(j,j+1) : \Delta_{K*\mu\, ij} \to T_{*\mu\, ij}$ is an isomorphism, and by Lemma 2.3, we have

$$(j,j+1)\circ\sigma_{i,j}\circ(j,j+1)\circ t\circ(j,j+1)\circ\sigma_{i,j}\circ(j,j+1) = (i,j) \text{ or } \mathrm{id}_{K^n}.$$

If $(j,j+1)\circ\sigma_{i,j}\circ(j,j+1)\circ t\circ(j,j+1)\circ\sigma_{i,j}\circ(j,j+1) = \mathrm{id}_{K^n}$, then $t = \mathrm{id}_{K^n}$. If $(j,j+1)\circ\sigma_{i,j}\circ(j,j+1)\circ t\circ(j,j+1)\circ\sigma_{i,j}\circ(j,j+1) = (i,j)$, then

$$\begin{aligned} t &= (j,j+1)\circ\sigma_{i,j}\circ(j,j+1)\circ(i,j)\circ(j,j+1)\circ\sigma_{i,j}\circ(j,j+1)\\ &= (j,j+1)\circ\sigma_{i,j}\circ(i,j+1)\circ\sigma_{i,j}\circ(j,j+1)\\ &= (j,j+1)\circ\sigma_{i,j+1}\circ(i,j+1)\circ(j,j+1)\\ &= \sigma_{i,j}\circ(i,j). \end{aligned}$$

Thus we have $t = \sigma_{i,j}\circ(i,j)$. $\square$

From Lemma 2.3 and Lemma 2.4, the universal covering map μ induces a local isomorphism

$$\mu_*^{[n]} : \mathrm{Blow}_{\Delta_{K*\mu}\cup T_{*\mu}} K^n_{*\mu}/H \to \mathrm{Blow}_{\Delta^n_{E*}} E^n_*/\mathcal{S}_n = E_*^{[n]}.$$

Here $\mathrm{Blow}_A B$ is the blow up of B along $A \subset B$.

Lemma 2.5. *For every $x \in E_*^{[n]}$, $|(\mu_*^{[n]})^{-1}(x)| = 2$.*

Proof. For $(x_i)_{i=1}^n \in \Delta^n_{E*}$ with $x_1 = x_2$, there are n elements $y_1,\ldots,y_n$ of K such that $y_1 = y_2$ and $\mu(y_i) = x_i$ for $1 \le i \le n$. Then

$$(\mu^n)^{-1}((x_i)_{i=1}^n)\cap K^n_{*\mu} = \{y_1,\sigma(y_1)\}\times\cdots\times\{y_n,\sigma(y_n)\}.$$

For $\sigma_{i_1\ldots i_k} \in G$, since H is generated by $\mathcal{S}_n$ and $\sigma_{i_1\ldots i_k}$, if k is even we get $\sigma_{i_1\ldots i_k} \in H$, if k is odd $\sigma_{i_1\ldots i_k} \notin H$. For $\{z_i\}_{i=1}^n \in (\mu^n)^{-1}((x_i)_{i=1}^n)\cap K^n_{*\mu}$, if the number of i with $z_i = y_i$ is even then

$$\{z_i\}_{i=1}^n = \{\sigma(y_1),\sigma(y_2),y_3\ldots,y_n\} \text{ on } K^n_{*\mu}/H, \text{ and}$$

if the number of i with $z_i = y_i$ is odd then

$$\{z_i\}_{i=1}^n = \{\sigma(y_1),y_2,y_3\ldots,y_n\} \text{ on } K^n_{*\mu}/H.$$

Furthermore since $\sigma_i \notin H$ for $1 \le i \le n$,

$$\{\sigma(y_1),\sigma(y_2),y_3\ldots,y_n\} \ne \{\sigma(y_1),y_2,y_3\ldots,y_n\}.$$

Thus for every $x \in E_*^{[n]}$, $|(\mu_*^{[n]})^{-1}(x)| = 2$. $\square$

Proposition 2.6. $\mu_*^{[n]} : \mathrm{Blow}_{\Delta_{K*\mu}\cup T_{*\mu}} K^n_{*\mu}/H \to \mathrm{Blow}_{\Delta^n_{E*}} E^n_*/\mathcal{S}_n$ *is the universal covering space, and* $X \setminus \pi^{-1}(F_E) \simeq \mathrm{Blow}_{\Delta_{K*\mu}\cup T_{*\mu}} K^n_{*\mu}/H$.

Proof. Since $\mu_*^{[n]}$ is a local isomorphism and the number of fiber is constant, so $\mu_*^{[n]}$ is a covering map. Furthermore $\pi : X \setminus \pi^{-1}(F_E) \to E_*^{[n]}$ is the universal covering space and number of fiber is 2, so $\mu_*^{[n]} : \mathrm{Blow}_{\Delta_{K*\mu}\cup T_{*\mu}} K^n_{*\mu}/H \to \mathrm{Blow}_{\Delta^n_{E*}} E^n_*/\mathcal{S}_n$ is the universal covering space, and by the uniqueness of the universal covering space, we have $X \setminus \pi^{-1}(F_E) \simeq \mathrm{Blow}_{\Delta_{K*\mu}\cup T_{*\mu}} K^n_{*\mu}/H$. $\square$

Recall that H is generated by $\mathcal{S}_n$ and $\{\sigma_{ij}\}_{1\le i<j\le n}$.

Theorem 2.7. *Let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$, and $n \ge 2$. Then there is a resolution $\varphi_X : X \to K^n/H$ such that $\varphi_X^{-1}(\Gamma_K/H) = \pi^{-1}(D_E)$.*

Proof. Let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$ where X is a Calabi-Yau manifold, and ρ the covering involution of π. From Proposition2.6, we have $X \setminus \pi^{-1}(F_E) \simeq \mathrm{Blow}_{\Delta_{K*\mu}\cup T_{*\mu}} K^n_{*\mu}/H$. Thus there is a meromorphim f of X to K^n/H with satisfying the following commutative diagram:

$$\begin{tikzcd} E^{[n]} \setminus F_E \arrow[r, "\pi_E"] & E^{(n)} \\ X \setminus \pi^{-1}(F_E) \arrow[u, "\pi"] \arrow[r, "f"] & K^n/H \arrow[u, "p_H"'] \end{tikzcd}$$

where $\pi_E : E^{[n]} \to E^{(n)}$ is the Hilbert-Chow morphism, and $p_H : K^n/H \to E^{(n)}$ is the natural projection. For any ample line bundle $\mathcal{L}$ on $E^{(n)}$, since the natural projection $p_H : K^n/H \to E^{(n)}$ is finite, and $E^{(n)}$ and K^n/H are projective, $p_H^*\mathcal{L}$ is ample. Since $\pi^{-1}(F_E)$ is an analytic closed subset of codimension 2 in X, there is a line bundle $\mathbb{L}$ on X such that $f^*(p_H^*\mathcal{L}) = \mathbb{L}\mid_{X\setminus\pi^{-1}(F_E)}$. From the above diagram, we have

$$\mathbb{L} = \pi^*(\pi_E^*\mathcal{L}).$$

Since $\mathcal{L}$ is ample on $E^{(n)}$, $\pi_E^*\mathcal{L}$ is a globally generated line bundle on $E^{[n]}$. Moreover $\pi^*(\pi_E^*\mathcal{L})$ is also a globally generated line bundle on X. Since $p_H^*\mathcal{L}$ is ample on K^n/H and $\mathbb{L}$ is globally generated, there is a holomorphism φ_X of X to K^n/H such that $\varphi_X\mid_{X\setminus\pi^{-1}(F_E)} = f\mid_{X\setminus\pi^{-1}(F_E)}$. Since X is a proper and the image of f contains a Zariski open subset, $\varphi_X : X \to K^n/H$ is surjective. Moreover $f : X \setminus \pi^{-1}(D_E) \cong (K^n \setminus \Gamma_K)/H$, that is a resolution. $\square$

3. Proof of Theorem 1.1

Let S be a smooth projective surface and $P(n)$ the set of partitions of n. We write $\alpha \in P(n)$ as $\alpha = (\alpha_1, \dots, \alpha_n)$ with $1\cdot\alpha_1 + \cdots + n\cdot\alpha_n = n$, and put $|\alpha| := \sum_i \alpha_i$. We put $S^\alpha := S^{\alpha_1}\times\cdots\times S^{\alpha_n}$, $S^{(\alpha)} := S^{(\alpha_1)}\times\cdots\times S^{(\alpha_n)}$ and $S^{[n]}$ the Hilbert scheme of n points of S. The cycle type $\alpha(g)$ of $g \in \mathcal{S}_n$ is the partition $(1^{\alpha_1(g)}, \dots, n^{\alpha_n(g)})$ where $\alpha_i(g)$ is the number of cycles with length i as the representation of g in a product of disjoint cycles. As usual, we denote by $(n_1, \dots, n_r)$ the cycle defined by mapping n_i to n_{i+1} for $i < r$ and n_r to n_1. By Steenbrink [11, page 526-530], $S^{(\alpha)}$ $(\alpha \in P(n))$ have the Hodge decomposition. By Göttsche and Soergel [4, Theorem 2], we have an isomorphism of Hodge structures:

$$H^{i+2n}(S^{[n]},\mathbb{C})(n) = \sum_{\alpha\in P(n)} H^{i+2|\alpha|}(S^{(\alpha)},\mathbb{C})(|\alpha|)$$

where $H^{i+2|\alpha|}(S^{(\alpha)},\mathbb{C})(|\alpha|)$ is the Tate twist of $H^{i+2|\alpha|}(S^{(\alpha)},\mathbb{C})$,
and $H^{i+2n}(S^{[n]},\mathbb{C})(n)$ is the Tate twist of $H^{i+2n}(S^{[n]},\mathbb{C})$. Since $H^{i+2n}(S^{[n]},\mathbb{C})(n)$ is a Hodge structure of weight $i + 2n - 2n = i$, we have $H^{i+2n}(S^{[n]},\mathbb{C})(n)^{p,q} = H^{i+2n}(S^{[n]},\mathbb{Q})^{p+n,q+n}$ for $p, q \in \mathbb{Z}$ with $p + q = i$, and $H^{i+2|\alpha|}(S^{(\alpha)},\mathbb{C})(|\alpha|)$ is a

Hodge structure of weight $i+2|\alpha|-2|\alpha|=i$, we have $H^{i+2|\alpha|}(S^{(\alpha)},\mathbb{C})(|\alpha|)^{p,q} = H^{i+2|\alpha|}(S^{(\alpha)},\mathbb{C})^{p+|\alpha|,q+|\alpha|}$ for $p,q\in\mathbb{Z}$ with $p+q=i$. Thus we have

$$\dim_{\mathbb{C}} H^{2n}(S^{[n]},\mathbb{C})^{1,2n-1} = \sum_{\alpha\in P(n)} \dim_{\mathbb{C}} H^{2|\alpha|}(S^{(\alpha)},\mathbb{C})^{1-n+|\alpha|,n-1+|\alpha|}. \tag{1}$$

Let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and $\pi : X\to E^{[n]}$ the universal covering space of $E^{[n]}$ where X is a Calabi-Yau manifold.

Proposition 3.1. $\dim_{\mathbb{C}} H^1(E^{[n]},\Omega^{2n-1}_{E^{[n]}})=0$.

Proof. From [11, page 526-530], $E^{(n)}$ have the Hodge decomposition, we have

$$H^{2n}(E^{[n]},\mathbb{C})^{1,2n-1}\simeq H^{2n-1}(E^{[n]},\Omega^1_{E^{[n]}}),\ \text{and}$$
$$H^{2n}(E^{(n)},\mathbb{C})^{1,2n-1}\simeq H^{2n-1}(E^{(n)},\Omega^1_{E^{(n)}}).$$

Similarly since $E^{(\alpha)}$ ($\alpha\in P(n)$) has the Hodge decomposition, if $1-n+|\alpha|<0$ or $n-1+|\alpha|>2n$ for $\alpha\in P(n)$, then

$$H^{2|\alpha|}(E^{(\alpha)},\mathbb{C})(|\alpha|)^{1-n+|\alpha|,n-1+|\alpha|}=0.$$

For $\alpha\in P(n)$ with $1-n+|\alpha|\geq 0$ and $n-1+|\alpha|\leq 2n$, then $|\alpha|=n-1$, $|\alpha|=n$ or $|\alpha|=n+1$. By the definition of $\alpha\in P(n)$ and $|\alpha|$, we obtain $\alpha = \{(n,0,\dots,0),(n-2,1,0,\dots,0)\}$. Thus, by the above equation (1), we have

$$\dim_{\mathbb{C}} H^{2n}(E^{[n]},\mathbb{C})^{1,2n-1} = \dim_{\mathbb{C}} H^{2n}(E^{(n)},\mathbb{C})^{1,2n-1}\oplus H^{2n-2}(E^{(n-2)}\times E^{(2)},\mathbb{C})^{0,2n-2}.$$

From the Künneth Theorem, we obtain

$$H^{2n-2}(E^{(n-2)}\times E^{(2)},\mathbb{C})^{0,2n-2}\simeq \bigoplus_{s+t=2n-2} H^s(E^{(n-2)},\mathbb{C})^{0,s}\otimes H^t(E^{(2)},\mathbb{C})^{0,t}.$$

Since $H^1(E,\mathbb{C})^{0,1}=H^2(E,\mathbb{C})^{0,2}=0$, we have

$$H^{2n-2}(E^{(n-2)}\times E^{(2)},\mathbb{C})^{0,2n-2}=0.$$

Let Λ be a subset of $\mathbb{Z}^{2n}_{\geq 0}$

$$\Lambda := \{(s_1,\cdots,s_n,t_1,\cdots,t_n)\in\mathbb{Z}^{2n}_{\geq 0} : \Sigma_{i=1}^n s_i = 1,\ \Sigma_{j=1}^n t_j = 2n-1\}.$$

From the Künneth Theorem, we have

$$H^{2n}(E^n,\mathbb{C})^{1,2n-1}\simeq \bigoplus_{(s_1,\cdots,s_n,t_1,\cdots,t_n)\in\Lambda}\left(\bigotimes_{i=1}^n H^2(E,\mathbb{C})^{s_i,t_i}\right).$$

Since $n\geq 2$, for each $(s_1,\cdots,s_n,t_1,\cdots,t_n)\in\Lambda$, there is a number $i\in\{1,\cdots,n\}$ such that $s_i=0$. Thus since $H^2(E,\mathbb{C})^{0,2}=0$, we have $H^{2n-1}(E^n,\mathbb{C})^{1,2n-1}=0$, so $H^{2n-1}(E^{(n)},\mathbb{C})^{1,2n-1}=0$. Hence $H^1(E^{[n]},\Omega^{2n-1}_{E^{[n]}})=0$. □

Theorem 3.2. *Let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and X the universal covering space of $E^{[n]}$. Then all small deformations of X is induced by that of $E^{[n]}$.*

Proof. Since each canonical bundle of E and $E^{[n]}$ is torsion, and from Ran [10, Corollary 2], they have unobstructed deformations. The Kuranishi family of E has a 10-dimensional smooth base, so the Kuranishi family of $E^{[n]}$ has a 10-dimensional smooth base by [3, Theorems 0.1 and 0.3]. Thus we have $\dim_{\mathbb{C}} H^1(E^n,T_{E^{[n]}})=10$.

Since $K_{E^{[n]}}$ is not trivial and $2K_{E^{[n]}}$ is trivial, we have

$$T_{E^{[n]}}\simeq \Omega^{2n-1}_{E^{[n]}}\otimes K_{E^{[n]}}.$$

Therefore we have $\dim_{\mathbb{C}} H^1(E^n, \Omega^{2n-1}_{E^{[n]}} \otimes K_{E^{[n]}}) = 10$. Since K_X is trivial, then we have $T_X \simeq \Omega^{2n-1}_X$. Since $\pi : X \to E^{[n]}$ is the covering map and

$$X \simeq \mathcal{S}pec\, \mathcal{O}_{E^{[n]}} \oplus \mathcal{O}_{E^{[n]}}(K_{E^{[n]}})$$

by [8, Theorem 3.1], we have

$$\begin{aligned} H^k(X, \Omega^{2n-1}_X) &\simeq H^k(E^{[n]}, \Omega^{2n-1}_{E^{[n]}} \oplus (\Omega^{2n-1}_{E^{[n]}} \otimes K_{E^{[n]}})) \\ &\simeq H^k(E^{[n]}, \Omega^{2n-1}_{E^{[n]}}) \oplus H^k(E^{[n]}, \Omega^{2n-1}_{E^{[n]}} \otimes K_{E^{[n]}}). \end{aligned}$$

Combining this with Proposition 3.1, we obtain

$$\dim_{\mathbb{C}} H^1(X, \Omega^{2n-1}_X) = \dim_{\mathbb{C}} H^1(E^{[n]}, \Omega^{2n-1}_{E^{[n]}} \otimes K_{E^{[n]}}).$$

Since $\pi : X \to E^{[n]}$ is a covering map, $\pi^* : H^1(E^{[n]}, T_{E^{[n]}}) \hookrightarrow H^1(X, T_X)$ is injective. Thus we have $\dim_{\mathbb{C}} H^1(X, T_X) = 10$.

Let $p : \mathcal{Y} \to U$ be the universal family of $E^{[n]}$ and $f : \mathcal{X} \to \mathcal{Y}$ is the universal covering space. Then $q : \mathcal{X} \to U$ is a flat family of X where $q := p \circ f$. Then we have a commutative diagram:

$$\begin{tikzcd} T_{U,0} \arrow[r, "\rho_p"] \arrow[dr, "\rho_q"'] & \mathrm{H}^1(\mathcal{Y}_0, T_{\mathcal{Y}_0}) \arrow[r, equal] \arrow[d, "\tau"] & H^1(E^{[n]}, T_{E^{[n]}}) \arrow[d, "\pi^*"] \\ & \mathrm{H}^1(\mathcal{X}_0, T_{\mathcal{X}_0}) \arrow[r, equal] & H^1(X, T_X). \end{tikzcd}$$

Since $H^1(E^{[n]}, T_{E^{[n]}}) \simeq H^1(X, T_X)$ by π^*, the vertical arrow τ is an isomorphism and

$$\dim_{\mathbb{C}} H^1(\mathcal{X}_u, T_{\mathcal{X}_u}) = \dim_{\mathbb{C}} H^1(\mathcal{X}_u, \Omega^{2n-1}_{\mathcal{X}_u})$$

is a constant for some neighborhood of $0 \in U$, it follows that $q : \mathcal{X} \to U$ is the complete family of $\mathcal{X}_0 = X$, therefore $q : \mathcal{X} \to U$ is the versal family of $\mathcal{X}_0 = X$. Thus every fibers of any small deformation of X is the universal covering of some the Hilbert scheme of n points of some Enriques surface. $\square$

4. Proof of Theorem 1.2

Let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$ where X is a Calabi-Yau manifold. At first, we show that for an automorphism f of $E^{[n]}$, $f(D_E) = D_E \Leftrightarrow f^*(\mathcal{O}_{E^{[n]}}(D_E)) = \mathcal{O}_{E^{[n]}}(D_E)$. Next, we show Theorem 1.2.

Proposition 4.1. $\dim_{\mathbb{C}} H^0(E^{[n]}, \mathcal{O}_{E^{[n]}}(D_E)) = 1$.

Proof. Since D_E is effective, we obtain $\dim_{\mathbb{C}} H^0(E^{[n]}, \mathcal{O}_{E^{[n]}}(D_E)) \geq 1$. Since the codimension of $\Delta^{(n)}_E$ is 2 in $E^{(n)}$, and $E^{(n)}$ is normal, we have

$$H^0(E^{(n)}, \mathcal{O}_{E^{(n)}}) = \Gamma(E^{(n)} \setminus \Delta^{(n)}_E, \mathcal{O}_{E^{(n)}}).$$

Since $\pi_E|_{E^{[n]} \setminus D_E} : E^{[n]} \setminus D_E \simeq E^{(n)} \setminus \Delta^{(n)}_E$, and $\mathcal{O}_{E^{[n]}}(D_E) \simeq \mathcal{O}_{E^{[n]}}$ on $E^{[n]} \setminus D_E$, we have

$$(\pi_E)_*(\mathcal{O}_{E^{[n]}}(D_E)) \simeq \mathcal{O}_{E^{(n)}} \text{ on } E^{(n)} \setminus \Delta^{(n)}_E.$$

Hence

$$\Gamma(E^{[n]} \setminus D_E, \mathcal{O}_{E^{[n]}}(D_E)) \simeq H^0(E^{(n)}, \mathcal{O}_{E^{(n)}}).$$

Since $E^{(n)}$ is compact, we have $H^0(E^{(n)}, \mathcal{O}_{E^{(n)}}) \simeq \mathbb{C}$. Therefore we have

$$\dim_{\mathbb{C}} \Gamma(E^{[n]} \setminus D_E, \mathcal{O}_{E^{[n]}}(D_E)) = 1.$$

Thus we obtain $\dim_{\mathbb{C}} H^0(E^{[n]}, \mathcal{O}_{E^{[n]}}(\pi^*(D_E))) = 1$. $\square$

Remark 4.2. Then by Proposition 4.1, for an automorphism $\varphi \in \mathrm{Aut}(E^{[n]})$, the condition $\varphi^*(\mathcal{O}_{E^{[n]}}(D_E)) = \mathcal{O}_{E^{[n]}}(D_E)$ is equivalent to the condition $\varphi(D_E) = D_E$.

Recall that $\pi \circ \omega : K^n \setminus \Gamma_K \to E^{[n]} \setminus D_E$ is the universal covering space.

Theorem 4.3. *Let E be an Enriques surface, D_E the exceptional divisor of the Hilbert-Chow morphism $\pi_E : E^{[n]} \to E^{(n)}$. An automorphism f of $E^{[n]}$ is natural if and only if $f(D_E) = D_E$, i.e. $f^*(\mathcal{O}_{E^{[n]}}(D_E)) = \mathcal{O}_{E^{[n]}}(D_E)$.*

Proof. Let f be an automorphism of $E^{[n]}$ with $f(D_E) = D_E$. Then f induces an automorphism of $E^{[n]} \setminus D_E$. Since the uniqueness of the universal covering space, there is an automorphism g of $K^n \setminus \Gamma_K$ such that $\pi \circ \omega \circ g = f \circ \pi \circ \omega$:

$$\begin{array}{ccc} K^n \setminus \Gamma_K & \xrightarrow{g} & K^n \setminus \Gamma_K \\ \downarrow{\scriptstyle \pi\circ\omega} & & \downarrow{\scriptstyle \pi\circ\omega} \\ E^{[n]} \setminus D_E & \xrightarrow{f} & E^{[n]} \setminus D_E. \end{array}$$

Since Γ_K is an analytic set of codimension 2, and K^n is projective, g can be extended to a birational automorphism of K^n. By Oguiso [7, Proposition 4.1], g is an automorphism of K^n, and there are some automorphisms $g_1, \ldots, g_n \in \mathrm{Aut}(K)$ and $s \in \mathcal{S}_n$ such that $g = s \circ g_1 \times \cdots \times g_n$. Since $\mathcal{S}_n \subset G$, we can assume that $g = g_1 \times \cdots \times g_n$.

Recall that we denote the covering transformation group of $\pi \circ \omega$ by:

$$G := \{g \in \mathrm{Aut}(K^n \setminus \Gamma_K) : \pi \circ \omega \circ g = \pi \circ \omega\}.$$

By Proposition 4.4 below, we have $g_i = g_1$ or $g_1 \circ \sigma$ for $1 \le i \le n$ and $g_1 \circ \sigma = \sigma \circ g_1$. We denote $g_1^{[n]}$ the induced automorphism of $E^{[n]}$ given by g_1. Then ${g_1}^{[n]}|_{E^{[n]} \setminus D_E} = f|_{E^{[n]} \setminus D_E}$. Thus $g_1^{[n]} = f$, so f is natural. The other implication is obvious. $\square$

Proposition 4.4. *In the proof of Theorem 4.3, we have $g_i = g_1$ or $g_i = g_1 \circ \sigma$ for each $1 \le i \le n$. Moreover $g_1 \circ \sigma = \sigma \circ g_1$.*

Proof. We show the first assertion by contradiction. Without loss of generality, we may assume that $g_2 \neq g_1$ and $g_2 \neq g_1 \circ \sigma$. Let h_1 and h_2 be two morphisms of K where $g_i \circ h_i = \mathrm{id}_K$ and $h_i \circ g_i = \mathrm{id}_K$ for $i = 1, 2$. We define two morphisms $H_{1,2}$ and $H_{1,2,\sigma}$ from K to K^2 by following.

$$H_{1,2} : K \ni x \mapsto (h_1(x), h_2(x)) \in K^2$$

$$H_{1,2,\sigma} : K \ni x \mapsto (h_1(x), \sigma \circ h_2(x)) \in K^2.$$

Let $S_\sigma := \{(x, y) | \, y = \sigma(x)\}$ be the subset of K^2. Since $h_1 \neq h_2$ and $h_1 \neq \sigma \circ h_2$, $H_{1,2}^{-1}(\Delta_K^2) \cup H_{1,2,\sigma}^{-1}(S_\sigma)$ do not coincide K. Thus there is $x' \in K$ such that $H_{1,2}(x') \notin \Delta_K^2$ and $H_{1,2,\sigma}(x') \notin S_\sigma$. For $x' \in K$, we put $x_i := h_i(x') \in K$ for $i = 1, 2$. Then there are some elements $x_3, \ldots, x_n \in K$ such that $(x_1, \ldots, x_n) \in K^n \setminus \Gamma_K$. We have $g((x_1, \ldots, x_n)) \notin K^n \setminus \Gamma_K$ by the assumption of x_1 and x_2. It is contradiction,

because g is an automorphism of $K^n\backslash\Gamma_K$. Thus we have $g_i = g_1$ or $g_i = g_1 \circ \sigma$ for $1 \leq i \leq n$.

We show the second assertion. Since the covering transformation group of $\pi \circ \omega$ is G, the liftings of f are given by

$$\{g \circ u : u \in G\} = \{u \circ g : u \in G\}.$$

Thus for $\sigma_1 \circ g$, there is an element $\sigma_{i_1 \cdots i_k} \circ s$ of G where $s \in \mathcal{S}_n$ and $t \in \{\sigma_{i_1 \ldots i_k}\}_{1 \leq k \leq n,\ 1 \leq i_1 < \ldots < i_k \leq n}$ such that $\sigma_1 \circ g = g \circ \sigma_{i_1 \cdots i_k} \circ s$. If we think about the first component of $\sigma_1 \circ g$ and [6, Lemma 1.2], we have $s = \mathrm{id}$ and $t = \sigma_1$. Therefore $g \circ \sigma_1 \circ g^{-1} = \sigma_1$, we have $\sigma \circ g_1 = g_1 \circ \sigma$. $\square$

5. Proof of Theorem 1.3

Let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, and $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$ where X is a Calabi-Yau manifold. First, for $n = 2$, we compute the Hode number of X. Next, for $n \geq 3$, we show that the covering involution of $\pi : X \to E^{[n]}$ acts on $H^2(X, \mathbb{C})$ as identity, and by using Theorem 1.2, we classify automorphisms of X acting on $H^2(X, \mathbb{C})$ identically and its order is 2. Finally, we show Theorem 1.3.

We suppose $n = 2$. Since $E^2_* = E^2$, we have $E^{[2]} = E^{[2]}_* = \mathrm{Blow}_{\Delta^2_E} E^2/\mathcal{S}_2$. Let $\pi : X \to E^{[2]}$ be the universal covering space of $E^{[2]}$. Since $K^2_{*\mu} = K^2$ and Proposition 2.6, we have

$$X \simeq \mathrm{Blow}_{\Delta^2_K \cup T} K^2/H,$$

where $T := \{(x, y) \in K^2 : y = \sigma(x)\}$. Let $\eta : \mathrm{Blow}_{\Delta^2_K \cup T} K^2/H \to K^2/H$ be the natural map. We put

$$D_\Delta := \eta^{-1}(\Delta^2_K/H) \text{ and}$$
$$D_T := \eta^{-1}(T/H).$$

For two inclusions

$$j_{D_\Delta} : D_\Delta \hookrightarrow \mathrm{Blow}_{\Delta^2_K \cup T} K^2/H, \text{ and}$$
$$j_{D_T} : D_T \hookrightarrow \mathrm{Blow}_{\Delta^2_K \cup T} K^2/H,$$

let j_{*D_Δ} be the Gysin morphism

$$j_{*D_\Delta} : H^p(D_\Delta, \mathbb{C}) \to H^{p+2}(\mathrm{Blow}_{\Delta^2_K \cup T} K^2/H, \mathbb{C}),$$

j_{*D_T} the Gysin morphism

$$j_{*D_T} : H^p(D_T, \mathbb{C}) \to H^{p+2}(\mathrm{Blow}_{\Delta^2_K \cup T} K^2/H, \mathbb{C}), \text{and}$$
$$\psi := \eta^* + j_{*D_\Delta} \circ \eta|^*_{D_\Delta} + j_{*D_T} \circ \eta|^*_{D_T}$$

morphisms from $H^p(K^2/H, \mathbb{C}) \oplus H^{p-2}(\Delta^2_K/H, \mathbb{C}) \oplus H^{p-2}(T/H, \mathbb{C})$ to $H^p(\mathrm{Blow}_{\Delta^2_K \cup T} K^2/H, \mathbb{C})$. From [12, Theorem 7.31], we have isomorphisms of Hodge structure on $H^k(\mathrm{Blow}_{\Delta^2_K \cup T} K^2/H, \mathbb{C})$ by ψ:

(2)
$$H^k(K^2/H, \mathbb{C}) \oplus H^{k-2}(\Delta^2_K/H, \mathbb{C}) \oplus H^{k-2}(T/H, \mathbb{C}) \simeq H^k(\mathrm{Blow}_{\Delta^2_K \cup T} K^2/H, \mathbb{C}).$$

For algebraic variety Y, let $h^{p,q}(Y)$ be the number $h^{p,q}(Y) = \dim_{\mathbb{C}} H^{p+q}(Y, \mathbb{C})^{p,q}$.

Theorem 5.1. *For the universal covering space $\pi : X \to E^{[2]}$, we have $h^{0,0}(X) = 1$, $h^{1,0}(X) = 0$, $h^{2,0}(X) = 0$, $h^{1,1}(X) = 12$, $h^{3,0}(X) = 0$, $h^{2,1}(X) = 0$, $h^{4,0}(X) = 1$, $h^{3,1}(X) = 10$, and $h^{2,2}(X) = 131$.*

Proof. Let σ be the covering involution of $\mu : K \to E$. Put

$$H^k_{\pm}(K,\mathbb{C})^{p,q} := \{\alpha \in H^k(K,\mathbb{C})^{p,q} : \sigma^*(\alpha) = \pm\alpha\} \text{ and}$$
$$h^{p,q}_{\pm}(K) := \dim_{\mathbb{C}} H^k_{\pm}(K,\mathbb{C})^{p,q}.$$

Then for an Enriques surface $E \simeq K/\langle\sigma\rangle$, we have

$$H^k(E,\mathbb{C})^{p,q} \simeq H^k_+(K,\mathbb{C})^{p,q}.$$

Since K is a $K3$ surface, we have

$$h^{0,0}(K) = 1,\ h^{1,0}(K) = 0,\ h^{2,0}(K) = 1,\ \text{and } h^{1,1}(K) = 20, \text{ and}$$
$$h^{0,0}_+(K) = 1,\ h^{1,0}_+(K) = 0,\ h^{2,0}_+(K) = 0, \text{ and}\, h^{1,1}_+(K) = 10, \text{ and}$$
$$h^{0,0}_-(K) = 0,\ h^{1,0}_-(K) = 0,\ h^{2,0}_-(K) = 1, \text{ and}\, h^{2,0}_-(K) = 10.$$

Since $n = 2$, we obtain $\Delta^2_K/H \simeq E$ and $T/H \simeq E$. Thus we have

$$h^{0,0}(\Delta^2_K/H) = 1,\ h^{1,0}(\Delta^2_K/H) = 0,\ h^{2,0}(\Delta^2_K/H) = 0, \text{ and}\, h^{1,1}(\Delta^2_K/H) = 10,$$

and we have

$$h^{0,0}(T/H) = 1,\ h^{1,0}(T/H) = 0,\ h^{2,0}(T/H) = 0, \text{ and}\, h^{1,1}(T/H) = 10.$$

By the definition of H, we obtain $H = \langle \mathcal{S}_2, \sigma_{1,2}\rangle$. From the Künneth Theorem, we have

$$H^{p+q}(K^2,\mathbb{C})^{p,q} \simeq \bigoplus_{s+u=p,t+v=q} H^{s+t}(K,\mathbb{C})^{s,t} \otimes H^{u+v}(K,\mathbb{C})^{u,v}, \text{ and}$$
$$H^k(K^2/H,\mathbb{C})^{p,q} \simeq \{\alpha \in H^k(K^2,\mathbb{C})^{p,q} : s^*(\alpha) = \alpha \,\text{for}\, s \in \mathcal{S}_2 \,\text{and}\, \sigma^*_{1,2}(\alpha) = \alpha\}.$$

Thus we obtain

$$h^{0,0}(K^2/H) = 1,\ h^{1,0}(K^2/H) = 0,\ h^{2,0}(K^2/H) = 0,\ h^{1,1}(K^2/H) = 10,$$
$$h^{3,0}(K^2/H) = 0,\ h^{2,1}(K^2/H) = 0,\ h^{4,0}(K^2/H) = 1,$$
$$h^{3,1}(K^2/H) = 10, \text{ and}\, h^{2,2}(K^2/H)^{2,2} = 111.$$

Specially, we fix a basis β of $H^2(K,\mathbb{C})^{2,0}$ and a basis $\{\gamma_i\}_{i=1}^{10}$ of $H^2_-(K,\mathbb{C})^{1,1}$, then we have

$$H^4(K^2/H,\mathbb{C})^{3,1} \simeq \bigoplus_{i=1}^{10} \mathbb{C}(\beta \otimes \gamma_i + \gamma_i \otimes \beta). \tag{3}$$

By the above equation (2), we have

$$h^{0,0}(\mathrm{Blow}_{\Delta^2_K \cup T} K^2/H) = 1,\ h^{1,0}(\mathrm{Blow}_{\Delta^2_K \cup T} K^2/H) = 0,$$
$$h^{2,0}(\mathrm{Blow}_{\Delta^2_K \cup T} K^2/H) = 0,\ h^{1,1}(\mathrm{Blow}_{\Delta^2_K \cup T} K^2/H) = 12,$$
$$h^{3,0}(\mathrm{Blow}_{\Delta^2_K \cup T} K^2/H) = 0,\ h^{2,1}(\mathrm{Blow}_{\Delta^2_K \cup T} K^2/H) = 0,$$
$$h^{4,0}(\mathrm{Blow}_{\Delta^2_K \cup T} K^2/H) = 1,\ h^{3,1}(\mathrm{Blow}_{\Delta^2_K \cup T} K^2/H) = 10, \text{ and}$$
$$h^{2,2}(\mathrm{Blow}_{\Delta^2_K \cup T} K^2/H) = 131.$$

Thus we obtain $h^{0,0}(X) = 1$, $h^{1,0}(X) = 0$, $h^{2,0}(X) = 0$, $h^{1,1}(X) = 12$, $h^{3,0}(X) = 0$, $h^{2,1}(X) = 0$, $h^{4,0}(X) = 1$, $h^{3,1}(X) = 10$, and $h^{2,2}(X) = 131$. □

We show that for $n \geq 3$, the covering involution of $\pi : X \to E^{[n]}$ acts on $H^2(X,\mathbb{C})$ as identity, by using Theorem 1.2 we classify automorphisms of X acting on $H^2(X,\mathbb{C})$ identically and its order is 2, and Theorem 1.3 from here.

Lemma 5.2. *Let X be a smooth complex manifold, $Z \subset X$ a closed submanifold with codimension is 2, $\tau : X_Z \to X$ the blow up of X along Z, $E = \tau^{-1}(Z)$ the exceptional divisor, and h the first Chern class of the line bundle $\mathcal{O}_{X_Z}(E)$.*
Then $\tau^* : H^2(X, \mathbb{C}) \to H^2(X_Z, \mathbb{C})$ is injective, and

$$H^2(X_Z, \mathbb{C}) \simeq H^2(X, \mathbb{C}) \oplus \mathbb{C}h.$$

Proof. Let $U := X \setminus Z$ be an open set of X. Then U is isomorphic to an open set $U' = X_Z \setminus E$ of X_Z. As τ gives a morphism between the pair (X_Z, U') and the pair (X, U), we have a morphism τ^* between the long exact sequence of cohomology relative to these pairs:

$$\begin{tikzcd}
H^k(X,U,\mathbb{C}) \arrow[r] \arrow[d, "\tau^*_{X,U}"] & H^k(X,\mathbb{C}) \arrow[r] \arrow[d, "\tau^*_X"] & H^k(U,\mathbb{C}) \arrow[r] \arrow[d, "\tau^*_U"] & H^{k+1}(X,U,\mathbb{C}) \arrow[d, "\tau^*_{X,U}"] \\
H^k(X_Z,U',\mathbb{C}) \arrow[r] & H^k(X_Z,\mathbb{C}) \arrow[r] & H^k(U',\mathbb{C}) \arrow[r] & H^{k+1}(X_Z,U',\mathbb{C}).
\end{tikzcd}$$

By Thom isomorphism, the tubular neighborhood Theorem, and Excision theorem, we have

$$H^q(Z, \mathbb{C}) \simeq H^{q+4}(X, U, \mathbb{C}), \text{ and}$$
$$H^q(E, \mathbb{C}) \simeq H^{q+2}(X_Z, U', \mathbb{C}).$$

In particular, we have

$$H^l(X, U, \mathbb{C}) = 0 \text{ for } l = 0, 1, 2, 3, \text{ and}$$
$$H^j(X_Z, U', \mathbb{C}) = 0 \text{ for } l = 0, 1.$$

Thus we have

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\tau^*_{X,U}"] & H^1(X,\mathbb{C}) \arrow[r] \arrow[d, "\tau^*_X"] & H^1(U,\mathbb{C}) \arrow[r] \arrow[d, "\tau^*_U"] & 0 \arrow[d, "\tau^*_{X,U}"] \\
0 \arrow[r] & H^1(X_Z,\mathbb{C}) \arrow[r] & H^1(U',\mathbb{C}) \arrow[r] & H^0(E,\mathbb{C}),
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\tau^*_{X,U}"] & H^2(X,\mathbb{C}) \arrow[r] \arrow[d, "\tau^*_X"] & H^2(U,\mathbb{C}) \arrow[r] \arrow[d, "\tau^*_U"] & 0 \arrow[d, "\tau^*_{X,U}"] \\
H^0(E,\mathbb{C}) \arrow[r] & H^2(X_Z,\mathbb{C}) \arrow[r] & H^2(U',\mathbb{C}) \arrow[r] & H^3(X_Z,U',\mathbb{C}).
\end{tikzcd}$$

Since $\tau \mid_{U'}: U' \xrightarrow{\sim} U$, we have isomorphisms $\tau_U^* : H^k(U, \mathbb{C}) \simeq H^k(U', \mathbb{C})$. Thus we have

$$\dim_{\mathbb{C}} H^2(X_Z, \mathbb{C}) = \dim_{\mathbb{C}} H^2(X, \mathbb{C}) + 1, \text{ and}$$
$$\tau^* : H^2(X, \mathbb{C}) \to H^2(X_Z, \mathbb{C}) \text{ is injective,}$$

and therefore we obtain

$$H^2(X_Z, \mathbb{C}) \simeq H^2(X, \mathbb{C}) \oplus \mathbb{C}h.$$

□

Proposition 5.3. *Suppose $n \geq 3$. For the universal covering space $\pi : X \to E^{[n]}$, $\dim_{\mathbb{C}} H^2(X, \mathbb{C}) = 11$.*

Proof. Since the codimension of $\pi^{-1}(F_E)$ is 2, $H^2(X,\mathbb{C}) \cong H^2(X \setminus \pi^{-1}(F_E),\mathbb{C})$.
By Proposition 2.6, $X \setminus \pi^{-1}(F_E) \simeq \mathrm{Blow}_{\Delta_{K*\mu}\cup T_{*\mu}} K^n_{*\mu}/H$.
Let $\tau : \mathrm{Blow}_{\Delta_{K*\mu}\cup T_{*\mu}} K^n_{*\mu} \to K^n_{*\mu}$ be the blow up of $K^n_{*\mu}$ along $\Delta_{K*\mu} \cup T_{*\mu}$,

h_{ij} the first Chern class of the line bundle $\mathcal{O}_{\mathrm{Blow}_{\Delta_{K*\mu}\cup T_{*\mu}} K^n_{*\mu}}(\tau^{-1}(\Delta_{K*\mu\, i,})),$

and

k_{ij} the first Chern class of the line bundle $\mathcal{O}_{\mathrm{Blow}_{\Delta_{K*\mu}\cup T_{*\mu}} K^n_{*\mu}}(\tau^{-1}(T_{K*\mu\, ij})).$

By Lemma 5.2, we have

$$H^2(\mathrm{Blow}_{\Delta_{K*\mu}\cup T_{*\mu}} K^n_{*\mu},\mathbb{C}) \cong H^2(K^n,\mathbb{C}) \oplus \left(\bigoplus_{1\le i<j\le n} \mathbb{C}h_{ij}\right) \oplus \left(\bigoplus_{1\le i<j\le n} \mathbb{C}k_{ij}\right).$$

Since $n \geq 3$, there is an isomorphism

$$(j,j+1)\circ\sigma_{ij}\circ(j,j+1) : \triangle_{K*\mu\, ij} \xrightarrow{\sim} T_{*\mu\, ij}.$$

Thus we have $\dim_{\mathbb{C}} H^2(\mathrm{Blow}_{\Delta_{K*\mu}\cup T_{*\mu}} K^n_{*\mu}/H,\mathbb{C}) = 11$, i.e. $\dim_{\mathbb{C}} H^2(X,\mathbb{C}) = 11$. □

Proposition 5.4. $\dim_{\mathbb{C}} H^0(X,\mathcal{O}_X(\pi^*(D_E))) = 1$.

Proof. Since π is finite, we obtain $\dim_{\mathbb{C}} H^0(X,\mathcal{O}_X(\pi^*(D_E))) = \dim_{\mathbb{C}} H^0(E^{[n]},\pi_*\mathcal{O}_X(\pi^*(D_E)))$.
From the projective formula and $X \simeq \mathcal{S}pec\, \mathcal{O}_{E^{[n]}} \oplus \mathcal{O}_{E^{[n]}}(K_{E^{[n]}})$, we have $\pi_*\mathcal{O}_X(\pi^*(D_E)) \simeq \mathcal{O}_{E^{[n]}}(D_E) \oplus \mathcal{O}_{E^{[n]}}(D_E \otimes K_{E^{[n]}})$. By Proposition 4.1, $\dim_{\mathbb{C}} H^0(E^{[n]},\mathcal{O}_{E^{[n]}}(D_E)) = 1$.
We show that

$$\dim_{\mathbb{C}} H^0(E^{[n]},\mathcal{O}_{E^{[n]}}(D_E \otimes K_{E^{[n]}})) = 0.$$

Since $\pi_E|_{E^{[n]}\setminus D_E} : E^{[n]} \setminus D_E \simeq E^{(n)} \setminus \Delta_E^{(n)}$, we have

$$(\pi_E)_*(\mathcal{O}_{E^{[n]}}(D_E \otimes K_{E^{[n]}})) \simeq \Omega^{2n}_{E^{(n)}} \text{ on } E^{(n)} \setminus \Delta_E^{(n)}.$$

Hence we have

$$\Gamma(E^{[n]} \setminus D_E, \mathcal{O}_{E^{[n]}}(D_E \otimes K_{E^{[n]}})) \simeq \Gamma(E^{(n)} \setminus \Delta_E^{(n)}, \Omega^{2n}_{E^{(n)}}).$$

Since $H^2(E,\mathbb{C})^{2,0} = 0$, and from the Künneth Theorem,

$$H^{2n}(E^n,\mathbb{C})^{2n,0} = H^0(E^n,\Omega^{2n}_{E^n}) = 0.$$

Since the codimension of Δ_E^n is 2, and $\Omega^{2n}_{E^n}$ is a locally free sheaf, we have

$$\Gamma(E^n \setminus \Delta_E^n, \Omega^{2n}_{E^n}) = H^0(E^n,\Omega^{2n}_{E^n}).$$

Thus we have

$$\Gamma(E^{(n)} \setminus \Delta_E^{(n)}, \Omega^{2n}_{E^{(n)}}) = 0,$$

and therefore

$$\dim_{\mathbb{C}} H^0(E^{[n]} \setminus D_E, \mathcal{O}_{E^{[n]}}(D_E \otimes K_{E^{[n]}})) = 0.$$

Hence

$$\dim_{\mathbb{C}} H^0(E^{[n]}, \mathcal{O}_{E^{[n]}}(D_E \otimes K_{E^{[n]}})) = 0.$$

Thus we obtain $\dim_{\mathbb{C}} H^0(X,\mathcal{O}_X(\pi^*(D_E))) = 1$. □

Remark 5.5. Then by Proposition 5.4, for an automorphism $\varphi \in \mathrm{Aut}$*(X),* the condition $\varphi^*(\mathcal{O}_X(\pi^* D_E)) = \mathcal{O}_X(\pi^* D_E)$ is equivalent to the condition $\varphi(\pi^{-1}(D_E)) = \pi^{-1}(D_E)$.

Let ρ be the covering involution of $\pi : X \to E^{[n]}$.

Proposition 5.6. *For $n \geq 3$, the induced map $\rho^* : H^2(X, \mathbb{C}) \to H^2(X, \mathbb{C})$ is identity.*

Proof. Since $E^{[n]} \simeq X/\langle \rho \rangle$, we have $H^2(E^{[n]}, \mathbb{C}) \simeq H^2(X, \mathbb{C})^{\rho^*}$. By Proposition 5.3, for $n \geq 3$, we have $\dim_{\mathbb{C}} H^2(X, \mathbb{C}) = 11$. By [1, page 767], $\dim_{\mathbb{C}} H^2(E^{[n]}, \mathbb{C}) = 11$. Thus the induced map $\rho^* : H^2(X, \mathbb{C}) \to H^2(X, \mathbb{C})$ is identity for $n \geq 3$. □

Recall that $\mu : K \to E$ is the universal covering of E where K is a $K3$ surface, and σ the covering involution of μ.

Proposition 5.7. *Let E be an Enriques surface which does not have numerically trivial involutions, $E^{[n]}$ the Hilbert scheme of n points of E, $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$, ρ the covering involution of π, and $n \geq 3$. Let ι be an involution of X which acts on $H^2(X, \mathbb{C})$ as* id, *then $\iota = \rho$.*

Proof. Let ι be an involution of X which acts on $H^2(X, \mathbb{C})$ as id. By Remark 5.5, $\iota|_{X \setminus \pi^{-1}(D_E)}$ is automorphism of $X \setminus \pi^{-1}(D_E)$. By the uniqueness of the universal covering space, there is an automorphism g of $K^n \setminus \Gamma_K$ such that $\iota \circ \omega = \omega \circ g$:

$$\begin{tikzcd} K^n \setminus \Gamma_K \arrow[r, "g"] \arrow[d, "\omega"] & K^n \setminus \Gamma_K \arrow[d, "\omega"] \\ X \setminus \pi^{-1}(D_E) \arrow[r, "\iota"] & X \setminus \pi^{-1}(D_E). \end{tikzcd}$$

Like the proof of Proposition 4.4, we can assume that there are some automorphisms g_i of K such that $g = g_1 \times \cdots \times g_n$, for each $1 \leq i \leq n$, $g_i = g_1$ or $g_i = g_1 \circ \sigma$, and $g_1 \circ \sigma = \sigma \circ g_1$. Since $\iota^2 = \mathrm{id}_X$, so we have $g^2 \in H$. Thus we have $g^2 = \mathrm{id}_{K^n}$ or $\sigma_{i_1 \ldots i_k}$. By [6, Lemma 1.2], we have $g^2 = \mathrm{id}_{K^n}$. We put $g' := g_1$. Let g'_E be the induced automorphism of E by g', and $g'^{[n]}_E$ the induced automorphism of $E^{[n]}$ by g'_E. Since $g'^{[n]}_E \circ \pi = \pi \circ \iota$ and $n \geq 3$, $g'^{[n]*}_E$ acts on $H^2(E^{[n]}, \mathbb{C})$ as id, and therefore g'^*_E acts on $H^2(E, \mathbb{C})$ as id. Since E does not have numerically trivial involutions, $g'_E = \mathrm{id}_E$, and therefore we have $g' = \sigma$ or $g' = \mathrm{id}_K$. Thus we have $\pi \circ \omega \circ g = \pi \circ \omega$:

$$\begin{tikzcd} K^n \setminus \Gamma_K \arrow[r, "g"] \arrow[d, "\pi\circ\omega"] & K^n \setminus \Gamma_K \arrow[d, "\pi\circ\omega"] \\ E^{[n]} \setminus D_E \arrow[r, "\mathrm{id}"] & E^{[n]} \setminus D_E. \end{tikzcd}$$

Since $\iota \circ \omega = \omega \circ g$, we have we have $\pi = \pi \circ \iota$:

$$\begin{tikzcd} X \setminus \pi^{-1}(D_E) \arrow[r, "\iota"] \arrow[d, "\pi"] & X \setminus \pi^{-1}(D_E) \arrow[d, "\pi"] \\ E^{[n]} \setminus D_E \arrow[r, "\mathrm{id}"] & E^{[n]} \setminus D_E. \end{tikzcd}$$

Since the degree of π is 2, we have $\iota = \rho$. □

We suppose that E has numerically trivial involutions. By [6, Proposition 1.1], there is just one automorphism of E, denoted υ, such that its order is 2, and υ^* acts on $H^2(E, \mathbb{C})$ as id. For υ, there are just two involutions of K which are liftings of υ, one acts on $H^0(K, \Omega^2_K)$ as id, and another acts on $H^0(K, \Omega^2_K)$ as $-$id, we denote by υ_+ and υ_-, respectively. Then they satisfies $\upsilon_+ = \upsilon_- \circ \sigma$. Let $\upsilon^{[n]}$

be the automorphism of $E^{[n]}$ which is induced by υ. For $\upsilon^{[n]}$, there are just two automorphisms of X which are liftings of $\upsilon^{[n]}$, denoted ς and ς', respectively:

$$\begin{CD} X @>{\varsigma\,(\varsigma')}>> X \\ @V{\pi}VV @VV{\pi}V \\ E^{[n]} @>{\upsilon^{[n]}}>> E^{[n]}. \end{CD}$$

Then they satisfies $\varsigma = \varsigma' \circ \sigma$. Since $n \geq 3$ and like the proof of Proposition 5.7, each order of ς and ς' is 2 .

Lemma 5.8. *For ς and ς', one acts on $H^0(X, \Omega_X^{2n})$ as* id*, and another act on $H^0(X, \Omega_X^{2n})$ as* $-$id*.*

Proof. Since $\upsilon^{[n]}|_{E^{[n]}\setminus D_E}$ is an automorphism of $E^{[n]} \setminus D_E$, and from the uniqueness of the universal covering space, there is an automorphism g of $K^n \setminus \Gamma_K$ such that $\upsilon^{[n]} \circ \pi \circ \omega = \pi \circ \omega \circ g$:

$$\begin{CD} K^n \setminus \Gamma_K @>{g}>> K^n \setminus \Gamma_K \\ @V{\pi\circ\omega}VV @VV{\pi\circ\omega}V \\ E^{[n]} \setminus D_E @>{\upsilon^{[n]}}>> E^{[n]} \setminus D_E. \end{CD}$$

Like the proof of Proposition 4.4, we can assume that there are some automorphisms g_i of K such that $g = g_1 \times \cdots \times g_n$ for each $1 \leq i \leq n$, $g_i = g_1$ or $g_i = g_1 \circ \sigma$, and $g_1 \circ \sigma = \sigma \circ g_1$. From Theorem 2.7, we get $K^n \setminus \Gamma_K/H \simeq X \setminus \pi^{-1}(D_E)$. Put

$$\upsilon_{+,even} := u_1 \times \cdots \times u_n$$

where

$$u_i = \upsilon_+ \text{ or } u_i = \upsilon_- \text{ and the number of } i \text{ with } u_i = \upsilon_+ \text{ is even}$$

which is an automorphism of K^n and induces an automorphism $\widetilde{\upsilon_{+,even}}$ of $X \setminus \pi^{-1}(D_E)$. We define automorphisms $\widetilde{\upsilon_{+,odd}}$, $\widetilde{\upsilon_{-,even}}$, and $\widetilde{\upsilon_{-,odd}}$ of $K^n \setminus \Gamma_K/H$ in the same way. Since $\sigma_{ij} \in H$ for $1 \leq i < j \leq n$, and $\upsilon_+ = \upsilon_- \circ \sigma$, if n is odd,

$$\widetilde{\upsilon_{+,odd}} = \widetilde{\upsilon_{-,even}},\ \widetilde{\upsilon_{+,even}} = \widetilde{\upsilon_{-,odd}}, \text{ and } \widetilde{\upsilon_{+,odd}} \neq \widetilde{\upsilon_{+,even}},$$

and if n is even,

$$\widetilde{\upsilon_{+,odd}} = \widetilde{\upsilon_{-,odd}},\ \widetilde{\upsilon_{+,even}} = \widetilde{\upsilon_{-,even}}, \text{ and } \widetilde{\upsilon_{+,odd}} \neq \widetilde{\upsilon_{+,even}}.$$

Since $\upsilon^{[n]} \circ \pi = \pi \circ \widetilde{\upsilon_{+,odd}}$ and $\upsilon^{[n]} \circ \pi = \pi \circ \widetilde{\upsilon_{+,even}}$, and the degree of π is 2, Thus we have $\{\varsigma, \varsigma'\} = \{\widetilde{\upsilon_{+,odd}}, \widetilde{\upsilon_{+,even}}\}$.

Let $\omega_X \in H^0(X, \Omega_X^{2n})$ be a basis of $H^0(X, \Omega_X^{2n})$ over $\mathbb{C}$. Since $X \setminus \pi^{-1}(F_E) \simeq \mathrm{Blow}_{\Delta_{K*\mu} \cup T_{*\mu}} K_{*\mu}^n/H$, and by the definition of υ_+ and υ_-,

$$\widetilde{\upsilon_{+,odd}}^*(\omega_X) = -\omega_X \text{ and } \widetilde{\upsilon_{+,even}}^*(\omega_X) = \omega_X.$$

Thus for $\{\varsigma, \varsigma'\}$, one acts on $H^0(X, \Omega_X^{2n})$ as id, and another act on $H^0(X, \Omega_X^{2n})$ as $-$id. $\square$

We put $\varsigma_+ \in \{\varsigma, \varsigma'\}$ as acts on $H^0(X, \Omega_X^{2n})$ as id and $\varsigma_- \in \{\varsigma, \varsigma'\}$ as acts on $H^0(X, \Omega_X^{2n})$ as $-$id.

Proposition 5.9. *Suppose E has numerically trivial involutions. Let $E^{[n]}$ be the Hilbert scheme of n points of E, $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$, ρ the covering involution of π, and $n \geq 3$. Let ι be an involution of X which ι^* acts on $H^2(X, \mathbb{C})$ as* id *and on $H^0(X, \Omega_X^{2n})$ as* $-$id*, and $\iota \neq \rho$. Then we have $\iota = \varsigma_-$.*

Proof. Let ι be an involution of X which acts on $H^2(X, \mathbb{C})$ as id and on $H^0(X, \Omega_X^{2n})$ as $-$id, and $\iota \neq \rho$. By Remark 5.5, $\iota|_{X \setminus \pi^{-1}(D_E)}$ is an automorphism of $X \setminus \pi^{-1}(D_E)$. By the uniqueness of the universal covering space, there is an automorphism g of $K^n \setminus \Gamma_K$ such that $\iota \circ \omega = \omega \circ g$:

$$\begin{array}{ccc} K^n \setminus \Gamma_K & \xrightarrow{g} & K^n \setminus \Gamma_K \\ \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \omega} \\ X \setminus \pi^{-1}(D_E) & \xrightarrow{\iota} & X \setminus \pi^{-1}(D_E). \end{array}$$

Like the proof of Proposition 4.4, we can assume that there are some automorphisms g_i of K such that $g = g_1 \times \cdots \times g_n$, for each $1 \leq i \leq n$, $g_i = g_1$ or $g_i = g_1 \circ \sigma$, and $g_1 \circ \sigma = \sigma \circ g_1$. Since $\iota^2 = \mathrm{id}_X$, so we have $g^2 \in H$. Thus we have $g^2 = \mathrm{id}_{K^n}$ or $\sigma_{i_1 \ldots i_k}$. By [6, Lemma 1.2], we have $g^2 = \mathrm{id}_{K^n}$. We put $g' := g_1$. Let g'_E be the induced automorphism of E by g', and $g'^{[n]}_E$ the induced automorphism of $E^{[n]}$ by g'_E. Since $g'^{[n]}_E \circ \pi = \pi \circ \iota$ and $n \geq 3$, $g'^{[n]*}_E$ acts on $H^2(E^{[n]}, \mathbb{C})$ as id, and therefore g'^*_E acts on $H^2(E, \mathbb{C})$ as *id*. If $g'_E = \mathrm{id}_E$, then we have $\iota = \rho$ or id_X, a contradiction. Since $g^2 = \mathrm{id}_{K^n}$ Thus the order of g'_E is 2. Since g'^*_E acts on $H^2(E, \mathbb{C})$ as id, we have $g'_E = \upsilon$, and therefore $g' = \upsilon_+$ or $g' = \upsilon_-$. By the definition of ς and ς', we obtain $\iota = \varsigma$ or $\iota = \varsigma'$. Since ι^* acts on $H^0(X, \Omega_X^{2n})$ as $-$id, we obtain $\iota = \varsigma_-$. □

Theorem 5.10. *Let E be an Enriques surface, $E^{[n]}$ the Hilbert scheme of n points of E, $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$, and $n \geq 3$. If X has a involution ι which ι^* acts on $H^2(X, \mathbb{C})$ as* id*, and $\iota \neq \rho$. Then E has a numerically trivial involution.*

Proof. Let ι be an involution of X which acts on $H^2(X, \mathbb{C})$ as id, and $\iota \neq \rho$. By Remark 5.5, $\iota|_{X \setminus \pi^{-1}(D_E)}$ is an automorphism of $X \setminus \pi^{-1}(D_E)$. By the uniqueness of the universal covering space, there is an automorphism g of $K^n \setminus \Gamma_K$ such that $\iota \circ \omega = \omega \circ g$:

$$\begin{array}{ccc} K^n \setminus \Gamma_K & \xrightarrow{g} & K^n \setminus \Gamma_K \\ \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \omega} \\ X \setminus \pi^{-1}(D_E) & \xrightarrow{\iota} & X \setminus \pi^{-1}(D_E). \end{array}$$

Like the proof of Proposition 4.4, we can assume that there are some automorphisms g_i of K such that $g = g_1 \times \cdots \times g_n$, for each $1 \leq i \leq n$, $g_i = g_1$ or $g_i = g_1 \circ \sigma$, and $g_1 \circ \sigma = \sigma \circ g_1$. Since $\iota^2 = \mathrm{id}_X$, we have $g^2 \in H$. Thus we have $g^2 = \mathrm{id}_{K^n}$ or $\sigma_{i_1 \ldots i_k}$. By [6, Lemma 1.2], we have $g^2 = \mathrm{id}_{K^n}$. We put $g' := g_1$. Let g'_E be the induced automorphism of E by g', and $g'^{[n]}_E$ the induced automorphism of $E^{[n]}$ by g'_E. Since $g'^{[n]}_E \circ \pi = \pi \circ \iota$ and $n \geq 3$, $g'^{[n]*}_E$ acts on $H^2(E^{[n]}, \mathbb{C})$ as id, and therefore g'^*_E acts on $H^2(E, \mathbb{C})$ as id. If $g'_E = \mathrm{id}$, like the proof of Proposition 5.7 we have $\iota = \rho$ or $\iota = \mathrm{id}_X$, a contradiction. Thus we have $g'_E \neq \mathrm{id}$. Since $g^2 = \mathrm{id}_{K^n}$, g'_E is an involution of E. Since g'^*_E acts on $H^2(E, \mathbb{C})$ as id, E has a numerically trivial involution. □

Lemma 5.11. *$dim_{\mathbb{C}} H^{2n-1,1}(K^n/H, \mathbb{C}) = 10$.*

Proof. Let σ be the covering involution of $\mu : K \to E$. Put

$$H^k_{\pm}(K, \mathbb{C})^{p,q} := \{\alpha \in H^k(K, \mathbb{C})^{p,q} : \sigma^*(\alpha) = \pm\alpha\} \text{ and}$$
$$h^{p,q}_{\pm}(K) := \dim_{\mathbb{C}} H^k_{\pm}(K, \mathbb{C})^{p,q}.$$

Since K is a $K3$ surface, we have

$$h^{0,0}(K) = 1,\ h^{1,0}(K) = 0,\ h^{2,0}(K) = 1, \text{ and } h^{1,1}(K) = 20, \text{ and}$$
$$h^{0,0}_{+}(K) = 1,\ h^{1,0}_{+}(K) = 0,\ h^{2,0}_{+}(K) = 0, \text{ and } h^{1,1}_{+}(K) = 10, \text{ and}$$
$$h^{0,0}_{-}(K) = 0,\ h^{1,0}_{-}(K) = 0,\ h^{2,0}_{-}(K) = 1, \text{ and } h^{2,0}_{-}(K) = 10.$$

Let Λ be a subset of $\mathbb{Z}^{2n}_{\geq 0}$

$$\Lambda := \{(s_1, \cdots, s_n, t_1, \cdots, t_n) \in \mathbb{Z}^{2n}_{\geq 0} : \Sigma^n_{i=1} s_i = 2n-1,\ \Sigma^n_{j=1} t_j = 1\}.$$

From the Künneth Theorem, we have

$$H^{2n}(k^n, \mathbb{C})^{2n-1,1} \simeq \bigoplus_{(s_1,\cdots,s_n,t_1,\cdots,t_n)\in\Lambda} \left(\bigotimes_{i=1}^{n} H^2(K, \mathbb{C})^{s_i,t_i} \right).$$

We fix a basis α of $H^2(K, \mathbb{C})^{2,0}$ and a basis $\{\beta_i\}_{i=1}^{10}$ of $H^2_{-}(K, \mathbb{C})^{1,1}$, and let

$$\tilde{\beta}_i := \bigotimes_{j=1}^{n} \epsilon_j$$

where $\epsilon_j = \alpha$ for $j \neq i$ and $\epsilon_j = \beta_i$ for $j = i$, and

$$\gamma_i := \bigoplus_{j=1}^{n} \tilde{\beta}_j.$$

then we have

$$H^{2n}(K^n/H, \mathbb{C})^{2n-1,1} \simeq \bigoplus_{i=1}^{10} \mathbb{C}\gamma_i, \tag{4}$$

$\dim_{\mathbb{C}} H^{2n-1,1}(K^n/H, \mathbb{C}) = 10$. $\square$

Since X and K^n/H are projective, K^n/H is a V-manifold, and π is a surjective, $\pi^* : H^{p,q}(K^n/H, \mathbb{C}) \to H^{p,q}(X, \mathbb{C})$ is injective.

Theorem 5.12. *We suppose $n \geq 2$. Let $\pi : X \to E^{[n]}$ be the universal covering space. For any automorphism f of X, if f^* is acts on $H^*(X, \mathbb{C}) := \bigoplus_{i=0}^{2n} H^i(X, \mathbb{C})$ as identity, then $f = \mathrm{id}_X$.*

Proof. Since f^* acts on $H^2(X, \mathbb{C})$ as identity, f is an automorphism of $K^n \setminus \Gamma_K/H$. Let $p_H : K^2 \setminus \Gamma_K \to K^2 \setminus \Gamma_K/H$ be the natural map. Then the uniqueness of the universal covering space, we can that there are some automorphisms g_i of K such that $g := g_1 \times \cdots \times g_n$, $g_i = g_1$ or $g_i = g_1 \circ \sigma$, $g_1 \circ \sigma = \sigma \circ g_1$ for $2 \leq i \leq n$, and $f \circ p_H = p_H \circ g$:

$$\begin{tikzcd}
K^n \setminus \Gamma_K/H \arrow[r, "f"] & K^n \setminus \Gamma_K/H \\
K^n \setminus \Gamma_K \arrow[u, "p_H"] \arrow[r, "g"] & K^n \setminus \Gamma_K. \arrow[u, "p_H"]
\end{tikzcd}$$

Let g_H be the induced automorphism of K^n/H. Then we obtain $g_H \circ \varphi_X = \varphi_X \circ f$:

$$\begin{array}{ccc} K^n/H & \xrightarrow{g_H} & K^n/H \\ {\scriptstyle\varphi_X}\uparrow & & {\scriptstyle\varphi_X}\uparrow \\ X & \xrightarrow{f} & X. \end{array}$$

Put g_{1E} the automorphism of E induced by g_1. Since f^* acts on $H^2(X,\mathbb{C})$ as identity, g_H^* acts on $H^2(K^n/H,\mathbb{C})$ as identity. Since $H^2(K^n/H,\mathbb{C}) \cong H^2(E,\mathbb{C})$, g_{1E}^* acts on $H^2(E,\mathbb{C})$ as identity. From Lemma 5.11, we have

$$H^{2n}(X,\mathbb{C})^{2n-1,1} = \bigoplus_{i=1}^{10} \mathbb{C}\varphi_X^*\gamma_i.$$

Suppose $g_1 \neq \sigma$ and $g_1 \neq \mathrm{id}_K$. Since g_{1E}^* acts on $H^2(E,\mathbb{C})$ as identity, from [6, page 386-389], the order of g_{1E} is at most 4. If the order of g_{1E} is 2, there is an element $\alpha_\pm \in H^2_-(K,\mathbb{C})^{1,1}$ such that $g_1^*(\alpha_\pm) = \pm\alpha$. By the equation (4) and the proof of Lemma 5.8, f does not act on $H^{2n}(X,\mathbb{C})^{2n-1,1}$ as identity, it is a contradiction. If the order of g_{1E} is 4, then there is an element $\alpha'_\pm \in H^2_-(K,\mathbb{C})^{1,1}$ such that $g_1^*(\alpha'_\pm) = \pm\sqrt{-1}\alpha'_\pm$ from [6, page 390-391]. By the equation (4) and and the proof of Lemma 5.8, f does not act on $H^{2n}(X,\mathbb{C})^{2n-1,1}$ as identity, it is a contradiction. Thus we have $g_{1E} = \mathrm{id}_E$, i.e. $g_1 = \sigma$ or $g_1 = \mathrm{id}_K$, and $f = \mathrm{id}_X$ or $f = \rho$ where ρ is the covering involution of $\pi : X \to E^n$. From Proposition 3.1 $H^{2n}(E^{[n]},\mathbb{C})^{2n-1,1} \simeq 0$, ρ does not act on $H^{2n}(X,\mathbb{C})^{2n-1,1}$ as identity. Since f^* acts on $H^{2n}(X,\mathbb{C})^{2n-1,1}$ as identity, we have $f = \mathrm{id}_X$. □

Corollary 5.13. *We suppose $n \geq 2$. Let $\pi : X \to E^{[2]}$ be the universal covering space. For any two automorphisms f and g of X, if $f^* = g^*$ on $H^*(X,\mathbb{C})$, then $f = g$.*

By [6, Proposition 1.1], there is just one automorphism of E, denoted υ, such that its order is 2, and υ^* acts on $H^2(E,\mathbb{C})$ as id. For υ, there are just two involutions of K which are liftings of υ, one acts on $H^0(K,\Omega_K^2)$ as id, and another acts on $H^0(K,\Omega_K^2)$ as $-$id, we denote by υ_+ and υ_-, respectively. Then they satisfies $\upsilon_+ = \upsilon_- \circ \sigma$. Let $\upsilon^{[n]}$ be the automorphism of $E^{[n]}$ which is induced by υ. For $\upsilon^{[n]}$, there are just two automorphisms of X which are liftings of $\upsilon^{[n]}$, denoted ς and ς', respectively. Then they satisfies $\varsigma = \varsigma' \circ \sigma$, and each order of ς and ς' is 2. From Lemma5.11, one acts on $H^0(X,\Omega_X^{2n})$ as id, and another act on $H^0(X,\Omega_X^{2n})$ as $-$id. We put $\varsigma_+ \in \{\varsigma,\varsigma'\}$ as acts on $H^0(X,\Omega_X^{2n})$ as id and $\varsigma_- \in \{\varsigma,\varsigma'\}$ as acts on $H^0(X,\Omega_X^{2n})$ as $-$id.

Theorem 5.14. *Let E and E' be two Enriques surfaces, $E^{[n]}$ and $E'^{[n]}$ the Hilbert scheme of n points of E and E', X and X' the universal covering space of $E^{[n]}$ and $E'^{[n]}$, and $n \geq 3$. If $X \cong X'$, then $E^{[n]} \cong E'^{[n]}$, i.e. when we fix X, then there is just one isomorphism class of the Hilbert schemes of n points of Enriques surfaces such that they have it as the universal covering space.*

Proof. For an involution of X which is the covering involution of some the Hilbert scheme of n points of Enriques surfaces acts on $H^2(X,\mathbb{C})$ as id, $H^0(X,\Omega_X^{2n})$ as $-$id, and $H^{2n}(X,\mathbb{C})^{2n-1,1}$ as $-$id. From Proposition5.12, the automorphisms which acts on $H^2(X,\mathbb{C})$ as id, $H^0(X,\Omega_X^{2n})$ as $-$id, are only ρ and ς_-. From the definition of

ς_- and Lemma 5.11, ς_- does not act on $H^{2n}(X,\mathbb{C})^{2n-1,1}$ as $-\mathrm{id}$. Thus we have an argument. $\square$

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(Taro Hayashi) Department of Mathematics, Graduate School of Science, Osaka University, Machikaneyamacho 1-1, Toyonaka, Osaka 560-0043, Japan

E-mail address: tarou-hayashi@cr.math.sci.osaka-u.ac.jp